

Agent-time Epistemics and Coordination

Ido Ben-Zvi

Department of Electrical Engineering, Technion
idobz@cs.technion.ac.il

Yoram Moses

Department of Electrical Engineering, Technion
moses@ee.technion.ac.il

March 1, 2013

Abstract

A minor change to the standard epistemic logical language, replacing K_i with $K_{\langle i, t \rangle}$ where t is a time instance, gives rise to a generalized and more expressive form of knowledge and common knowledge operators. We investigate the communication structures that are necessary for such generalized epistemic states to arise, and the inter-agent coordination tasks that require such knowledge. Previous work has established a relation between linear event ordering and nested knowledge, and between simultaneous event occurrences and common knowledge. In the new, extended, formalism, epistemic necessity is decoupled from temporal necessity. Nested knowledge and event ordering are shown to be related even when the nesting order does not match the temporal order of occurrence. The generalized form of common knowledge does *not* correspond to simultaneity. Rather, it corresponds to a notion of tight coordination, of which simultaneity is an instance.

1 Introduction

We have recently embarked on an in-depth inquiry concerning the relation between knowledge, coordination and communication in multi agent systems [5, 6, 4]. This study uncovered new structural connections between the three in systems where agents have accurate clocks, and there are (commonly known) bounds on the time it may take messages to be delivered, between any two neighboring agents. We call this the *synchronous* model. In such a setting, one often reasons about what agents know at particular

time points in the past or future, as well as, in particular, what they know about what other agents will know (or have known) at various other times, etc.

The emphasis on information regarding the times at which facts are known has lead us to consider a formalization in which epistemic operators are indexed by a pair consisting of an agent *and a time*—thus, $K_{\langle i, t \rangle}$ refers to what i knows at time t —rather than the more traditional epistemic operator K_i , in which knowledge is associated with an agent, and the formulas are evaluated with respect to a particular time. An agent-time pair $\langle i, t \rangle$ is called a *node*, and we distinguish the new *node-based* (or *nb-*) language from the traditional *agent-based* one.

In this paper we formulate an expressive language with nb-knowledge operators, and use it to extend and strictly strengthen the previously established relations between knowledge, coordination and communication. Our earlier inquiries reduced coordination tasks to states of knowledge, and then analyzed the communication requirements required to obtain these states of knowledge. Together these provide a crisp structural characterization of necessary and sufficient conditions for particular coordination tasks, in terms of the communication required.

We specify coordination tasks in terms of orchestrating a pattern of responses in reaction to a spontaneous external input initiated by the environment. Such a spontaneous event is considered as the *trigger* for its responses. As shown in [5], ensuring a linearly ordered sequence of responses to such a triggering event requires a state of deeply nested knowledge, in which the last responder to know that the next-to-last responder knows, . . . , that the agent performing the first response knows that the trigger event occurred. In *asynchronous* systems without clocks, such nested knowledge can only be obtained via a message chain from the trigger passing through the responders in their order in the sequence. The situation in the synchronous setting, in the presence of timing information, is much more delicate and interesting. In that case, such nested knowledge requires a particular pattern of communication called a *centipede*. While nested knowledge is captured by a centipede, common knowledge corresponds to a more restrictive and instructive communication pattern called a *broom*¹). Using the connection between common knowledge and simultaneity, brooms are shown to underly natural forms of simultaneous coordination.

The node-based operators give rise to natural strict generalizations of nested and common knowledge. Of particular interest is node-based com-

¹A structure called *centibroom* in [5] has since been renamed to be a broom.

mon knowledge, which is represented by an epistemic operator C_A , in which A is a set of agent-time pairs $\langle i, t \rangle$. So $C_{\{\langle i, t \rangle, \langle j, t+3 \rangle\}}\varphi$ indicates, among other things, that agent i at time t knows that agent j at time $t+3$ will know that i at time t knew that φ holds. While nb-common knowledge strictly generalizes classical common knowledge, it shares most of the typical attributes associated with common knowledge. However, whereas (traditional) common knowledge is intimately related to simultaneity, nb-common knowledge *is not!* We will show that, instead, nb-common knowledge precisely captures a form of coordination we call tightly-timed response (TTR). Of course, simultaneity is a particular form of tightly-timed coordination.

In general, we follow the pattern of investigation that we established in earlier papers: Given an epistemic state (such as nb-nested or nb-common knowledge) we examine, on the one hand, what form of communication among the agents is necessary for achieving such a state, and on the other we characterize the type of coordination tasks that require such a state. As nb-knowledge operators generalize the standard agent-based ones, the results we obtain here generalize and extend the ones of [5, 4]. Relating knowledge to communication, we present two “knowledge-gain” theorems (formulated in the spirit of Chandy and Misra’s result for systems without clocks [7]). Nb-nested and nb-common knowledge are shown to require more flexible variants of the centipede and the broom communication structures (respectively), in order to arise.

Perhaps more instructive is the characterization of the coordinative tasks related to these epistemic states. As mentioned above, nested knowledge characterizes agents that are engaged in responding in a sequential order to a triggering event. It turns out that Nb-nested knowledge is similarly necessary when the ordering task predefines not only the sequence of responses, but also a bound on the elapsed time between each response and its successor. We define this task as the *Weakly Timed Response* problem (WTR). Interestingly, minute differences in the coordinative task specification result in dramatic changes to the required epistemic state. As for common knowledge, as hinted above it will be seen that the node-based version relaxes the typical simultaneous responses requirement into tightly-timed ones.

The main contributions of this paper are:

- Node-based epistemic operators are defined. Under a natural semantic definition, both nb-knowledge and nb-common knowledge are S5 operators, while nb-common knowledge satisfies fixed-point and induction properties analogous to those of standard common knowledge.
- The theory of coordination and its relation to epistemic logic are ex-

tended, significantly generalizing previous results. Nb-semantics helps to decouple epistemic and temporal necessity. Namely, there are cases in which knowledge about (the guarantee of) someone’s knowledge at a future time, typically based on known communication, is required in order to perform an earlier action.

- The new epistemic operators are used as a formal tool for the study of generalized forms of coordination and the communication structures they require. On the one hand, natural coordination tasks that require nb-nested and nb-common knowledge are identified. On the other hand, the communication structures necessary for attaining these nb-epistemic states are established. Combining the two types of results yields new structural connection between communication and coordination tasks.
- The well-known strong connection between common knowledge and simultaneity established in [12, 11, 10] is shown to be a particular instance of a more general phenomenon: We prove that tightly-timed synchronization is very closely related to node-based common knowledge. The form of tight synchronization corresponding to agent-based common knowledge is precisely simultaneous coordination.

This paper is structured as follows. In the next section we define the syntax and semantics of two languages: One is a traditional agent-based logic of knowledge, and the other a node-based logic. An embedding of agent-based formulas in the nb-language is established, and basic properties of nb-knowledge and nb-ck are discussed. Section 3 presents the synchronous model within which we will study knowledge and coordination. In Section 4 a review of the notions underlying the earlier analysis, and its main results, is presented. Our new analysis is presented in Sections 5 and 6. Section 5 defines “uneven” centipedes and relates them to nested nb-knowledge. It then defines a coordination task called *weakly-timed response* and shows that it is captured by nested nb-knowledge. Analogously, Section 6 defines uneven brooms and relates them to nested nb-common knowledge. It then defines *tightly-timed response* and shows that it is captured by nested nb-ck. Finally, a short discussion and concluding remarks are presented in Section 7. Proofs of the claims in Sections 5 and 6 are presented in the Appendix.

2 Agent and Node-based Semantics

We consider both the standard, agent-based, epistemic language and its extension into the node-based variant within the *interpreted systems* framework of [11]. In this framework, the multi agent system is viewed as consisting of a set $\mathbb{P} = \{1, \dots, n\}$ of agents, connected by a communication network which serves as the exclusive means by which the agents interact with each other.

We assume that, at any given point in time, each agent in the system is in some *local state*. A *global state* is just a tuple $g = \langle \ell_e, \ell_1, \dots, \ell_n \rangle$ consisting of local states of the agents, together with the state ℓ_e of the *environment*. The environment's state accounts for everything that is relevant to the system that is not contained in the state of the agents. A *run* r is a function from time to global states. Intuitively, a run is a complete description of what happens over time in one possible execution of the system. We use $r_i(t)$ to denote agent i 's local state ℓ_i at time t in run r , for $i = 1, \dots, n$. For simplicity, time here is taken to range over the natural numbers rather than the reals (so that time is viewed as discrete, rather than dense or continuous). *Round* t in run r occurs between time $t - 1$ and t . A *system* \mathcal{R} is an exhaustive set of all possible runs, given the agents' protocol and the *context*, where the latter determines underlying characteristics of the environment as a whole.

To reason about the knowledge states of agents, a simple logical language is introduced. Since we are focusing on coordination tasks in which actions are triggered by spontaneous events, the only primitive propositions we consider are ones that state that an event has occurred. The standard, agent-based, variant of this language is \mathcal{L}_0 . It uses the following grammar, with the usual abbreviations for \vee and \Rightarrow , and with e, i and G used as terms for an event, an agent and a group of agents, respectively.

$$\varphi ::= \text{occurred}(e) \mid \varphi \wedge \varphi \mid \neg \varphi \mid K_i \varphi \mid E_G \varphi \mid C_G \varphi$$

The semantics is as follows. Propositional connectives, omitted from the list, are given their usual semantics.

Definition 1 (\mathcal{L}_0 semantics). *The truth of a formula $\varphi \in \mathcal{L}_0$ is defined with respect to a triple (\mathcal{R}, r, t) .*

- $(\mathcal{R}, r, t) \models \text{occurred}(e)$ iff event e has occurred in r by time t .
- $(\mathcal{R}, r, t) \models K_i \varphi$ iff $(\mathcal{R}, r', t) \models \varphi$ for all $r \in \mathcal{R}$ s.t. $r_i(t) = r'_i(t)$.
- $(\mathcal{R}, r, t) \models E_G \varphi$ iff $(\mathcal{R}, r, t) \models K_i \varphi$ for every $i \in G$.

- $(\mathcal{R}, r, t) \models C_G \varphi$ iff $(\mathcal{R}, r, t) \models (E_G)^k \varphi$ for every $k \geq 1$.

Note that while non-epistemic formulas require only a run r and time t in order to be evaluated, the epistemic operators also require the complete system of runs \mathcal{R} (which provides the analogue of the set of states, or *possible worlds* in standard modal logic). The second clause, giving semantics for the knowledge operator K_i , has built-in an assumption that time is common knowledge, since knowledge at time t in r depends only on truth in other runs, that are indistinguishable for agent i from r at *the same time* t . This is appropriate for our intended analysis, since in our model (to be presented in Section 3) time is assumed to be common knowledge. We used \mathcal{L}_0 to formalize the analysis in [5].

We next define the node-based language \mathcal{L}_1 . It will be convenient to define the set $\mathbb{V} = \{\langle i, t \rangle : i \in \mathbb{P}, t \text{ is a time}\}$ of all possible nodes. In the grammar we now use e, α and A as terms for an event, an agent-time node and a group of such nodes, respectively.

$$\varphi ::= \text{occurred}_t(e) \mid \varphi \wedge \varphi \mid \neg \varphi \mid K_\alpha \varphi \mid E_A \varphi \mid C_A \varphi$$

Notice that all statements in this language are “time-stamped:” they refer to explicit times at which the stated facts hold. As a result, they are actually time-invariant, and state facts about the run, rather than facts whose truth depends on the time of evaluation. Therefore, semantics for formulas of \mathcal{L}_1 are given with respect to a system \mathcal{R} and a run $r \in \mathcal{R}$. The semantics is as follows (again, omitting propositional clauses).

Definition 2 (\mathcal{L}_1 semantics). *The truth of a formula $\varphi \in \mathcal{L}_1$ is defined with respect to a pair (\mathcal{R}, r) .*

- $(\mathcal{R}, r) \models \text{occurred}_t(e)$ iff event e has occurred in r by time t
- $(\mathcal{R}, r) \models K_{\langle i, t \rangle} \varphi$ iff $(\mathcal{R}, r') \models \varphi$ for all runs r' s.t. $r_i(t) = r'_i(t)$.
- $(\mathcal{R}, r) \models E_A \varphi$ iff $(\mathcal{R}, r) \models K_\alpha \varphi$ for every $\alpha \in A$.
- $(\mathcal{R}, r) \models C_A \varphi$ iff $(\mathcal{R}, r) \models (E_A)^k \varphi$ for every $k \geq 1$.

In principle, the node-based semantics, as proposed here, can be seen as a simplified version of a real-time temporal logic with an explicit clock variable [1], and more generally of a hybrid logic [2]. We are not aware of instances in which such logics have been combined with epistemic operators. Nevertheless, in this paper we aim to utilize the formalism, rather than explore it. Hence, issues of expressibility, completeness and tractability are left unattended, to be explored at a future date.

We now demonstrate that, in a precise sense, the traditional language \mathcal{L}_0 can be embedded in the language \mathcal{L}_1 , in a meaning-preserving manner. We do this by way of defining a “*timestamping*” operation ts transforming a formula $\varphi \in \mathcal{L}_0$ and a time t to a formula $\text{ts}(\varphi, t) = \varphi^t \in \mathcal{L}_1$ that is timestamped by t . We then prove the following lemma showing that the timestamping is sound.

Lemma 1. *There exists a function $\text{ts} : \mathcal{L}_0 \times \text{Time} \rightarrow \mathcal{L}_1$ such that for every $\varphi \in \mathcal{L}_0$, time t , and $\varphi^t = \text{ts}(\varphi, t)$:*

$$(\mathcal{R}, r, t) \models \varphi \quad \text{iff} \quad (\mathcal{R}, r) \models \varphi^t.$$

Lemma 1 shows that \mathcal{L}_1 is at least as expressive as \mathcal{L}_0 . Yet \mathcal{L}_1 is not equivalent in expressive power to \mathcal{L}_0 , since it allows for multiple temporal reference points where epistemic operators are involved. For example, the formula $K_{\langle i, t \rangle} K_{\langle j, t' \rangle} \varphi$ cannot be translated into an equivalent \mathcal{L}_0 formula, since \mathcal{L}_0 does not allow for the temporal reference point to be shifted when switching from the outer knowledge operator to the inner one.

Compare a typical agent-based nested knowledge formula, such as $\psi = K_i K_j \varphi$, with a node based counterpart such as $\psi' = K_{\langle i, t \rangle} K_{\langle j, t+3 \rangle} \varphi$. In ψ , gaining knowledge about the epistemic state of other agents means that the knowledge of the referred other agent has already been gained. By contrast, in ψ' agent i has epistemic certainty concerning the knowledge state of another agent j , at a particular time in the future. To sum up, the node-based formalism allows us to differentiate between epistemic necessity, and temporal tense. Formally trivial, this decoupling is quite elusive as we tend to mix one with the other. The rest of this paper follows up on the implications of this point.

It is well-known that standard agent-based common knowledge is closely related to simultaneity [12, 11, 10]. Indeed, both $C_G \varphi \Rightarrow E_G C_G \varphi$ and $K_i C_G \varphi \Rightarrow C_G \varphi$ are valid formulas. (Recall that a formula $\psi \in \mathcal{L}_0$ is *valid* if $(\mathcal{R}, r, t) \models \psi$ for all choices of \mathcal{R} , run $r \in \mathcal{R}$ and time t .) Thus, the first instant at which $C_G \varphi$ holds must involve a simultaneous change in the local states of all members of G . In contrast, simultaneity is *not* an intrinsic property of node-based common knowledge. As an example, consider the node set $A = \{\langle i, t \rangle, \langle j, t + 10 \rangle\}$. Although we still have that both $K_{\langle i, t \rangle} C_A \varphi \Rightarrow C_A \varphi$ and $C_A \varphi \Rightarrow K_{\langle j, t+10 \rangle} C_A \varphi$ are valid formulas, if the current time is t and i knows that $C_A \varphi$ holds, this does not mean that j currently knows this too. The bond with simultaneity has been broken. As we shall see in Section 6, however, a notion of tight temporal coordination that generalizes simultaneity *is* intrinsic to node-based common knowledge.

In the spirit of the treatment in [11], for a formula $\psi \in \mathcal{L}_1$ we write $\mathcal{R} \models \psi$ and say that ψ is *valid in* (the system) \mathcal{R} if $(\mathcal{R}, r) \models \psi$ for all $r \in \mathcal{R}$. Node-based common knowledge manifests many of the logical properties shown by the standard notion of common knowledge. Proof of the following lemma, and of all new lemmas in this paper, can be found in the Appendix.

Lemma 2.

- Both $K_{\langle i, t \rangle}$ and C_A are S5 modalities.
- $C_A \varphi \Rightarrow E_A(\varphi \wedge C_A \varphi)$ is valid.
- If $\mathcal{R} \models \varphi \Rightarrow E_A(\varphi \wedge \psi)$ then $\mathcal{R} \models \varphi \Rightarrow C_A \psi$

The second clause of Lemma 2 corresponds to the so-called “fixed-point” axiom of common knowledge, while the third clause corresponds to the “induction (inference) rule” [11].

3 The Synchronous Model

As mentioned, we focus on synchronous systems where the clocks of the individual agents are all synchronized, and there are commonly known bounds on message delivery times. Generally, in order to define a system of runs that conforms to the required setting of interest, we define the system of runs \mathcal{R} as a function $\mathcal{R} = \mathcal{R}(P, \gamma)$ of the protocol P followed by the agents, and the underlying context of use γ .

We identify a *protocol* for an agent i with a function from local states of i to nonempty sets of actions. (In this paper we assume a *deterministic* protocol, in which a local state is mapped to a singleton set of actions. Such a protocol essentially maps local states to actions.) A *joint protocol* is just a sequence of protocols $P = (P_1, \dots, P_n)$, one for each agent.

In this paper we will assume a specified protocol for the agents, namely the *fip*, or *full information protocol*. In this protocol, every agent sends out its complete history to each of its neighbors, on every round. In order to be able to do that, the agent must be able to recall its own history. We therefore also assume this capability, called *perfect recall*, for the agents. Note that using a pre-specified protocol is not in general necessary, and is done for deductive purposes only. Our findings can be applied to all protocols under slight modifications.

In order for a well defined system of runs \mathcal{R} to emerge, the context γ needs to be rigorously defined as well. We also assume a specific context

γ^{\max} , within which the agents are operating. Most notably, γ^{\max} specifies that (a) agents share a universal notion of time, (b) the communication network has upper bounds on delivery times, and (c) all nondeterminism is deferred to the environment agent.

A word regarding nondeterminism, that plays a crucial role in the analysis of knowledge gain. Formal analysis allows us to escape the more difficult questions associated with this concept. For us, intuitively, spontaneous events are ones that cannot be foretold by the agents in the system. These could stand for a power shortage, but also for some user input that is communicated to an agent via a console. Formally, since all of the agents are following a deterministic protocol, nondeterminism is only introduced into the system by the environment.

For brevity, we must omit the formal details², and make do with describing the properties of the outcome system $\mathcal{R} = \mathcal{R}(\text{fip}, \gamma^{\max})$:

- Global clock and global network - The current time is always common knowledge. The network, which can be encoded as a weighted graph, is also common knowledge. For each (i, j) connected by a communication channel, max_{ij} , the weight on the corresponding network edge, denotes the maximal transmission times for messages sent along this channel.
- Events - There are four kinds of events: message send and receive events, internal calculations, and external inputs. Events occur at a single agent, within a single round. All are self explanatory except for the last type. External input events occur when a signal is received by an agent from “outside” the system. This could be user input, fate, and also - less dramatically - it could be used to signify the initial values for internal variables at the beginning of a run.
- Environment protocol - The environment is responsible for the occurrence of two of the event types: message deliveries and external inputs. (a) message deliveries - the protocol dictates that message deliveries will occur only for messages that have been sent and are still en route. Apart from that, the environment nondeterministically chooses when to deliver sent messages, subject only to the constraint that for every communication channel (i, j) , transmission on the channel does not take longer than max_{ij} rounds. (b) external inputs - in each round the environment nondeterministically chooses a (possibly empty) set of agents at which external inputs will occur, and the kind of events

²The interested reader can find complete accounts of the context we assume in [5, 4]

that will occur there. The choice is entirely unconstrained by previous or simultaneous occurrences.

4 Previous Findings

Our approach is based on the findings of Lamport [13] and of Chandy and Misra [7]. In his seminal analysis, Lamport defines *potential causality*, a formalization of message chains. Two events are related by potential causality when the coordinates marking their occurrences (the time and agent at which they occur) are related by an unbroken sequence of messages. We rephrase the relation as one holding between pairs of agent-time nodes.

Definition 3 (Potential causality [13]). *Fix $r \in \mathcal{R}$. The potential causality relation \rightsquigarrow over nodes of r is the smallest relation satisfying the following three conditions:*

1. *If $t \leq t'$ then $\langle i, t \rangle \rightsquigarrow \langle i, t' \rangle$;*
2. *If some message is sent at $\langle i, t \rangle$ and received at $\langle j, t' \rangle$ then $\langle i, t \rangle \rightsquigarrow \langle j, t' \rangle$; and*
3. *If $\langle i, t \rangle \rightsquigarrow \langle h, t'' \rangle$ and $\langle h, t'' \rangle \rightsquigarrow \langle j, t' \rangle$, then $\langle i, t \rangle \rightsquigarrow \langle j, t' \rangle$.*

Lamport used the relation as a basis for his *logical clocks* mechanism that allows distributed protocol designers in asynchronous contexts to temporally order events despite the lack of synchronization. Chandy and Misra later gave an epistemic analysis, showing that for agent j at time t' to know of an occurrence at agent i 's at time t , it must be that $\langle i, t \rangle \rightsquigarrow \langle j, t' \rangle$ holds.

In [5, 6, 4] we applied similar methodology to analyze knowledge gain and temporal ordering of events in *synchronous* systems, like the one described here. Note that temporally ordering events in a system with global clock is quite easy if exact timing is prearranged as a part of the protocol:

Charlie will deposit the money at 3pm, you will sign the contract at 4, and I will deliver the merchandise at 5.

Coordination becomes more challenging once the time of occurrence of the triggering, initial, action is nondeterministic:

Charlie will deposit the money, err... whenever, then you will sign the contract, and only then will I deliver the merchandise.

To approach the issue of event ordering we define the *Ordered Response* coordination challenge. Such problems are defined based on a *triggering event* e_s , which is an event of type external input, and a set of *responses*

$\alpha_1, \dots, \alpha_k$. Each response α_h is a pair $\langle i_h, a_h \rangle$ indicating an action a_h that agent i_h is required to carry out. When a response α_h gets carried out in a specific run, we denote with $\alpha_h = \langle i_h, t_h \rangle$ the specific agent-time node at which action a_h occurs.

Definition 4 (Ordered Response [5]). *Let e_s be an external input nondeterministic event. A protocol P solves the instance $\text{OR} = \langle e_s, \alpha_1, \dots, \alpha_k \rangle$ of the Ordered Response problem if it guarantees that*

1. *every response α_h , for $h = 1, \dots, k$, occurs in a run iff the trigger event e_s occurs in that run.*
2. *for $h = 1, \dots, k - 1$, response α_h occurs no later than response α_{h+1} (i.e. $t_h \leq t_{h+1}$).*

We can show that solving the challenge for me, you and Charlie requires that you will know that Charlie has deposited before you sign the contract, and that I will know that you know that Charlie has made the deposit, before I deliver the merchandise. Theorem 1 below proves this link between coordination and knowledge. Like all of the results quoted from previous works in this section, it is expressed using the language \mathcal{L}_0 .

Theorem 1 ([4]). *Let $\text{OR} = \langle e_s, \alpha_1, \dots, \alpha_k \rangle$ be an instance of OR, and assume that OR is solved in the system \mathcal{R} . Let $r \in \mathcal{R}$ be a run in which e_s occurs, let $1 \leq h \leq k$, and let t_h be the time at which i_h performs action a_h in r . Then*

$$(\mathcal{R}, r, t_h) \models K_{i_h} K_{i_{h-1}} \dots K_{i_1} \text{occurred}(e_s).$$

Given that nested knowledge can be shown to be a necessary prerequisite for ordering actions, what patterns of communication will provide such knowledge gain? One way to ensure event ordering is to insist on a message chain, linking Charlie to you and then to me. In fact, as Chandy and Misra show, it is the *only* way to ensure this, in an asynchronous system. But the synchronous setting allows for more flexibility in ensuring event ordering. For example, we could have:

Charlie sends messages to both you and me, alerting us of the deposit. Given that transmission times are bounded from above, upon getting a message from Charlie I calculate how long before the message sent to you is guaranteed to arrive, and deliver the merchandise only then.

The extra flexibility, when compared to asynchronous systems, is based on the availability of guarantees on message delivery times. We formalize these guarantees in the following way:

Definition 5 (Bound guarantee [5]). *Fix $r \in \mathcal{R}$. The bound guarantee relation \dashrightarrow over nodes of r is the smallest relation satisfying the following three conditions:*

1. *If $t \leq t'$ then $\langle i, t \rangle \dashrightarrow \langle i, t' \rangle$;*
2. *If the network contains a channel (i, j) with weight \max_{ij} then $\langle i, t \rangle \dashrightarrow \langle j, t + \max_{ij} \rangle$; and*
3. *If $\langle i, t \rangle \dashrightarrow \langle h, t'' \rangle$ and $\langle h, t'' \rangle \dashrightarrow \langle j, t' \rangle$, then $\langle i, t \rangle \dashrightarrow \langle j, t' \rangle$.*

With most of the building blocks in place, we are almost ready to describe the Knowledge Gain Theorem, the technical result at the core of our [5] paper. The theorem characterizes the communication pattern that is necessary for nested knowledge gain in synchronous systems. Intuitively, it shows that message chains still play an important part in information flow, but the synchronous equivalent of a message chain is much more flexible - since here agents can also use bound guarantees in order to ensure that a message arrives at its destination. The resultant message chain abstraction, called the *centipede*, is defined below.

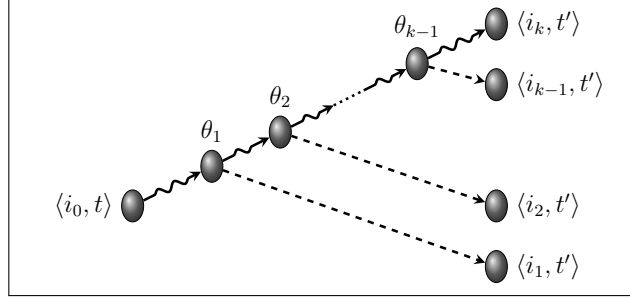


Figure 1: A centipede

Definition 6 (Centipede [5]). *Let $r \in \mathcal{R}$, let $i_h \in \mathbb{P}$ for $0 \leq h \leq k$ and let $t \leq t'$. A centipede for $\langle i_0, \dots, i_k \rangle$ in the interval $[t, t']$ in r is a sequence $\langle \theta_0, \theta_1, \dots, \theta_k \rangle$ of nodes such that $\theta_0 = \langle i_0, t \rangle$, $\theta_k = \langle i_k, t' \rangle$, $\theta_0 \rightsquigarrow \theta_1 \rightsquigarrow \dots \rightsquigarrow \theta_k$, and $\theta_h \dashrightarrow (i_h, t')$ holds for $h = 1, \dots, k - 1$.*

A centipede for $\langle i_0, \dots, i_k \rangle$ is depicted in Figure 1. It shows a message chain connecting (i_0, t) and (i_k, t') , and along this chain a sequence of “route splitting” nodes θ_1, θ_2 , etc. such that each θ_h can guarantee the arrival of a message to i_h by time t' . Such a message can serve to inform i_h of the

occurrence of a trigger event at $\langle i_0, t \rangle$, and as the set of previously made guarantees gets shuffled on to the next splitting node, the last agent i_k can be confident that by time t' all previous agents are already informed of the occurrence.

Theorem 2 (Knowledge Gain Theorem [5]). *Let $r \in \mathcal{R}$. Assume that e is an external event occurring at $\langle i_0, t \rangle$ in r .*

If $(\mathcal{R}, r, t') \models K_{i_k} K_{i_{k-1}} \cdots K_{i_0} \text{occurred}(e)$, then there is a centipede for $\langle i_0, \dots, i_k \rangle$ in the interval $[t, t']$ in r .

Theorem 1 states that solving the ordering problem requires nested knowledge of the occurrence of the trigger event to have been gained. Theorem 2 then shows that such knowledge gain can only take place if the agents are related by the centipede communication pattern.

Although stated in terms of our system \mathcal{R} , where agents are assumed to be following **fip**, under some further generalization both theorems can be made to apply for all systems based on the synchronous context γ^{\max} , regardless of the specific protocol.³

The ordering problem, nested knowledge and the centipede define a “vertical stack”, going from coordination, to knowledge, to communication and centered about nested knowledge gain. We now examine another such vertical stack, this time defined based on common knowledge. Common knowledge has been associated with simultaneous action already in [11, 12]. We touch upon this relation by defining the *Simultaneous Response* problem and then using Theorem 3 below.

Definition 7 (Simultaneous Response [5]). *Let e_s be an external input non-deterministic event. A protocol P solves the instance $\text{SR} = \langle e_s, \alpha_1, \dots, \alpha_k \rangle$ of the Simultaneous Response problem if it guarantees that*

1. *every response α_h , for $h = 1, \dots, k$, occurs in a run iff the trigger event e_s occurs in that run.*
2. *all of the responses $\alpha_1, \dots, \alpha_k$ are performed simultaneously (i.e. $t_1 = t_2 = \dots = t_k$).*

Theorem 3 ([5]). *Let $\text{SR} = \langle e_s, \alpha_1, \dots, \alpha_k \rangle$, and assume that SR is solved in \mathcal{R} . Moreover, let $G = \{i_1, \dots, i_k\}$ be the set of processes appearing in the response set of SR . Finally, let $r \in \mathcal{R}$ be a run in which e_s occurs, and let t be the time at which the response actions are performed in r . Then $(\mathcal{R}, r, t) \models C_G \text{occurred}(e_s)$.*

³ In order to generalize the theorems in this way we need to extend potential causality into a relation called *syncausality*. See [5] for more.

Relating common knowledge to a necessary communication pattern is more difficult, especially if you consider the formally-convenient classification of common knowledge as an infinite conjunction of nested mutual knowledge operators. Intuitively, borrowing the centipede from the nested knowledge results quotes above, this would mean that an infinite centipede is required in order for common knowledge to arise. However, it turns out that the necessary communication pattern is quite finite, and more in line with a fixed point view: In order for the group of agents G to gain common knowledge that e_s has occurred, there must exist a single agent-time node θ that is message chain related to the site of occurrence, and that can guarantee that forwarded messages with information regarding e_s to all members of G will arrive by t' . We call this pattern the *broom*. The following definition, and Figure 2 below, should make it apparent why.

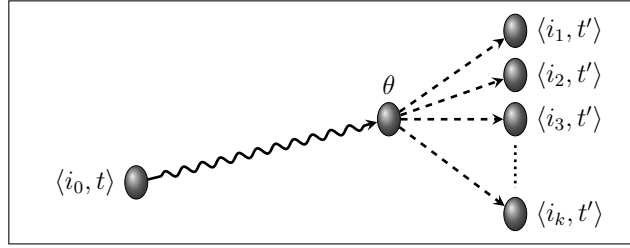


Figure 2: A broom

Definition 8 (Broom [5]). *Let $t \leq t'$ and $G \subseteq \mathbb{P}$. Node θ is a broom for $\langle i_0, G \rangle$ in the interval $[t, t']$ in r if $\langle i_0, t \rangle \rightsquigarrow \theta$ and $\theta \dashrightarrow \langle i_h, t' \rangle$ holds for all $i_h \in G$.*

Theorem 4 states the necessity connection between common knowledge gain and the broom structure. As before, when we were dealing with nested knowledge, Theorems 3 and 4 can be made to hold for all synchronous (γ^{\max} based) systems, regardless of protocol.

Theorem 4 (Common Knowledge Gain [5]). *Let $G \subseteq \mathbb{P}$, and let $r \in \mathcal{R}$. Assume that e is an external input event at $\langle i_0, t \rangle$ in r . If $(\mathcal{R}, r, t') \models C_G(\text{occurred}(e))$, then there is a broom $\hat{\theta}$ for $\langle i_0, G \rangle$ in interval $[t, t']$ in r .*

5 Nested Knowledge and Weak Bounds

We now re-approach the so-called “vertical stack” centered about nested knowledge. This time we replace standard nested formulas such as $K_i K_j K_k \varphi$

with nb-formulations such as $K_{\langle i, t \rangle} K_{\langle j, t-4 \rangle} K_{\langle k, t+8 \rangle} \varphi$.

What communication pattern is required in order to attain such nested knowledge? The following generalization of the centipede echos the above mentioned break between temporal precedence and necessity.

Definition 9 (Uneven centipede). *Let $r \in \mathcal{R}$, let $A = \langle \alpha_0, \alpha_1, \dots, \alpha_k \rangle$ be a sequence of nodes. An (uneven) centipede for A in r is a sequence $\langle \theta_0, \theta_1, \dots, \theta_k \rangle$ of nodes such that $\theta_0 = \alpha_0$, $\theta_k = \alpha_k$, $\theta_0 \rightsquigarrow \theta_1 \rightsquigarrow \dots \rightsquigarrow \theta_k$, and $\theta_h \dashrightarrow \alpha_h$ holds for $h = 1, \dots, k-1$.*

Figure 3 shows such an uneven centipede. It is termed *uneven* because the “legs” of the centipede end at nodes at a variety of different times, whereas in the traditional centipede all legs ended at nodes of time t_k . More interestingly, note that the node α_2 temporally precedes α_1 in Fig. 3. Intuitively though, this does not seem to concur with the epistemic status of the two nodes, because information about the occurrence of the trigger event at α_0 flows through θ_1 and θ_2 to α_2 , along with θ_1 ’s guarantee that by t_1 agent i_1 will have received word of the occurrence as well. Thus, $K_{\alpha_2} K_{\alpha_1} \text{occurred}_{t_0}(e_s)$ is attained by information flowing from a witness for α_1 to the node α_2 . Thus, in a precise sense, while α_1 occurs temporally later than α_2 , it “epistemically precedes” α_2 in this run.

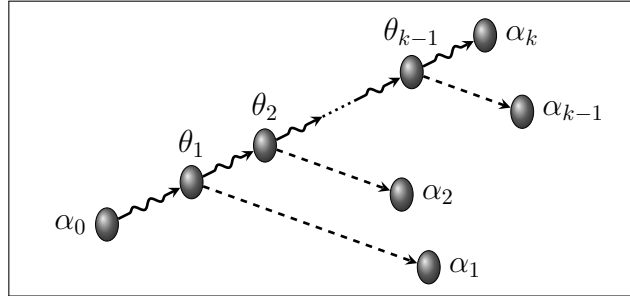


Figure 3: A uneven centipede

Theorem 5 shows that indeed the uneven centipede is necessary for nb-nested knowledge gain. Interestingly, the proof argument is identical to that which was used to prove the original Theorem 2. These proofs, originating in [5], can be found in the Appendix. The only thing changed is the formal language, and the proposed communication patterns. The same goes for Theorem 7 below, which utilizes the same proof as Theorem 4, with an update from \mathcal{L}_0 to \mathcal{L}_1 . It is a case where our understanding of what is going on has been limited purely by the expressivity of the formal apparatus of which we made use.

Theorem 5. *Let $r \in \mathcal{R}$. Assume that e_s is an external event occurring at $\alpha_0 = \langle i_0, t_0 \rangle$ in r .*

If $(\mathcal{R}, r) \models K_{\alpha_k} K_{\alpha_{k-1}} \dots K_{\alpha_0} \text{occurred}_{t_0}(e_s)$, then there is an uneven centipede for $\langle \alpha_0, \dots, \alpha_k \rangle$ in r .

We now turn to explore the implications of nb-nested knowledge for the coordinated Ordered Response problem. In line with our above discussion concerning nb-semantics and the decoupling of epistemic necessity and past tense, we expect that nb-nested knowledge will allow for greater flexibility in the timing of performed responses, solving tasks such as the following:

Once Charlie deposits the money, you will sign the contract, and I will deliver the merchandise no more than 5 days after you sign.

This suggests the OR variant $\langle \text{deposit}, \langle \text{You, sign} \rangle, \langle \text{I, deliver} \rangle \rangle$, where the required timing is such that $t_{\text{deliver}} \leq t_{\text{sign}} + 5$. Assume that Charlie, You and I act at nodes α_{Ch}, α_Y and α_I respectively. By modifying Theorem 1, we should be able to show that $K_{\alpha_I} K_{\alpha_Y} \text{occurred}_{t_Y}(\text{deposit})$ holds. Once we have shown this, Theorem 5 will show that the required communication is as shown in Figure 4a.

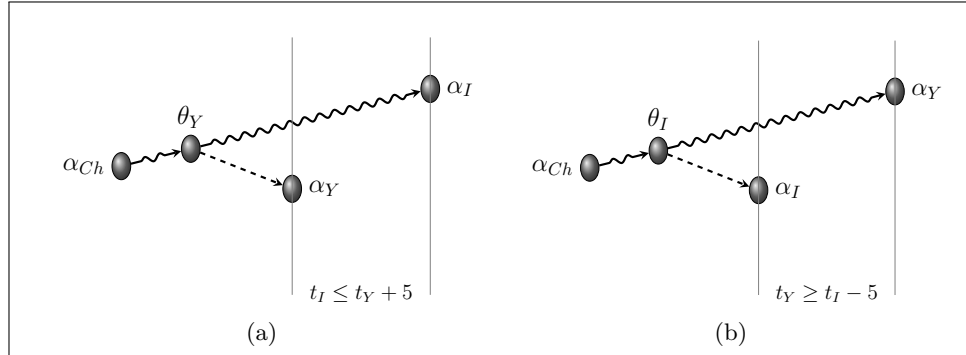


Figure 4

Perhaps surprisingly, it turns out that the required epistemic state for ensuring the correct execution of the problem as defined, is actually $K_{\alpha_Y} K_{\alpha_I} \text{occurred}_{t_I}(\text{deposit})$! Which in turn requires the inverted centipede shown in Figure 4b. To see why, consider that the requirement $t_{\text{deliver}} \leq t_{\text{sign}} + 5$ again. Seemingly an upper bound constraint upon *me* to deliver the merchandise no more than 5 days after contract sign, it can also be read as a lower bound constraint upon *You* to sign the contract no sooner than 5 days before delivery takes place... The point is, however, that *I* cannot promise to deliver the merchandise within the set time, since word of the signing

may arrive at my site a week after it occurs. Whereas *You*, no matter how late (or how soon) you hear of my having delivered the merchandise, can always wait a few days to ensure that the 5 days have passed.

The discussion concerning this timed coordination task points out a valuable insight. The knowledge that each agent must possess *as it responds*, concerns those responses whose time of occurrence is already bounded from above. These imply a lower bound for the agent's own action. In the first proposed (informal) definition, it is actually the case that *You* have knowledge of an upper limit upon *my* response. This brings about the reversal of the required nested knowledge, in comparison to what we expected it to be like. A task definition that exposes the implied *upper bounds* is in place.

We now provide a formal definition for the kind of coordination task that is characterized by nb-nested knowledge. The definition is phrased in accordance with the above discussion, and as Theorem 6 below shows, solutions to the problem indeed require that the intuitively proper nested knowledge holds. The required coordination is considered *weakly timed*, reflecting the added existence of timing constraints, which only bound from below. As we will see in the next section, the *tightly timed* task is similar, but contains stronger bounds on relative timing of responses.

Definition 10 (Weakly Timed Response). *Let e_s be an external input non-deterministic event. A protocol P solves the instance*

$\text{WTR} = \langle e_s, \alpha_1 : \delta_1, \alpha_2 : \delta_2, \dots, \alpha_{k-1} : \delta_{k-1}, \alpha_k \rangle$ *of the Weakly Timed Response problem if it guarantees that*

1. *every response α_h , for $h = 1, \dots, k$, occurs in a run iff the trigger event e_s occurs in that run.*
2. *in a run where response α_h occurs at $\alpha_h = \langle i_h, t_h \rangle$ for all $h \leq k$, for every such h we have that $t_{h+1} \geq t_h + \delta_h$.*

Before stating the theorem relating the WTR problem to nb-nested knowledge, another nuance should be observed. The problem definition specifies that even though agent i_k may not know the exact time at which responses are performed by other agents, it can work out an upper bound on the time of responses carried out by agents i_1 to i_{k-1} . For example, response α_{k-1} gets carried out at t_{k-1} which no later than $t_k - \delta_{k-1}$. Response α_{k-2} is then bounded with respect to α_k by $t_{k-2} \leq t_{k-1} - \delta_{k-2} \leq t_k - \delta_{k-1} - \delta_{k-2}$, etc. We will use

$$\beta_h^k = \langle i_h, t_h^k \rangle \quad \text{where } t_h^k = t_k - \sum_{j=h}^{k-1} \delta_j$$

to denote this upper limit: the latest possible node at which response α_h gets carried out, given that response α_k is performed at time t_k . Note that $t_h \leq t_h^k$ since by definition t_h^k is an upper bound on t_h . For the same reason we also have that $t_h^k \leq t_h^{k+1}$, since

$$t_h^{k+1} = t_{k+1} - \delta_k - \sum_{j=h}^{k-1} \delta_j = t_k^{k+1} - \sum_{j=h}^{k-1} \delta_j \geq t_k - \sum_{j=h}^{k-1} \delta_j = t_h^k.$$

Theorem 6. *Let $\text{WTR} = \langle e_s, \alpha_1 : \delta_1, \alpha_2 : \delta_2, \dots, \alpha_{k-1} : \delta_{k-1}, \alpha_k \rangle$ be an instance of WTR, and assume that WTR is solved in the system \mathcal{R} . Let $r \in \mathcal{R}$ be a run in which e_s occurs. For each $h \leq k$, let $\alpha_h = \langle i_h, t_h \rangle$ be the node at which response α_h gets performed, and let*

$$\beta_h^k = \langle i_h, t_k - \sum_{j=h}^{k-1} \delta_j \rangle.$$

Then

$$(\mathcal{R}, r) \models K_{\alpha_k} K_{\beta_{k-1}^k} \cdots K_{\beta_1^k} \text{occurred}_{t_1}(e_s).$$

6 Common Knowledge and Tight Bounds

In this section we examine nb-common knowledge and its relation to communication and coordination. Just as nb-nested knowledge requires an extension to the centipede structure, nb-common knowledge can only arise if the *uneven broom* communication structure, seen in Figure 5 and defined below, takes place in the run. (“Uneven” again comes from the broom’s uneven legs.)

Definition 11 (Uneven broom). *Let $A = \{\alpha_1, \dots, \alpha_k\}$ be a sequence of nodes and let α_0 be a node. Node θ is an (uneven) broom for $\langle \alpha_0, A \rangle$ in r if $\alpha_0 \rightsquigarrow \theta$ and $\theta \dashrightarrow \alpha_h$ holds for all $h = 1, \dots, k$.*

Theorem 7 proves that the uneven broom characterizes the necessary communication for nb-common knowledge to arise.

Theorem 7. *Let $A \subseteq \mathbb{V}$ with $\langle i_k, t_k \rangle$ being the latest node ($t_k \geq t_h$ for all $\langle i_h, t_h \rangle \in A$), and let $r \in \mathcal{R}$. Assume that e_s is an external input event at α_0 in r . If $(\mathcal{R}, r) \models C_A(\text{occurred}_{t_k}(e_s))$, then there is a uneven broom $\hat{\theta}$ for $\langle \alpha_0, A \rangle$ in r .*

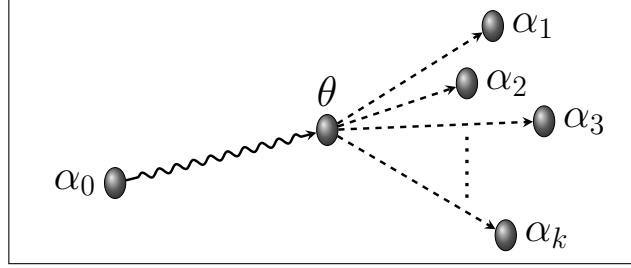


Figure 5: A uneven broom

Recall the Simultaneous Response problem, that shows standard common knowledge as the necessary requirement when a group of agents need to act simultaneously. Yet, as discussed in the introduction, node based common knowledge does away with simultaneity while retaining other properties of common knowledge. In accordance, we expect it to serve as the epistemic requirement for some weakened generalization of the simultaneity requirement. The following coordination task seems like a promising candidate.

Definition 12 (Tightly Timed Response). *Let e_s be an external input non-deterministic event. A protocol P solves the instance*

$$\text{TTR} = \langle e_s, \alpha_1 : \delta_1, \alpha_2 : \delta_2, \dots, \alpha_k : \delta_k \rangle$$

of the Tightly Timed Response problem if it guarantees that

1. *every response α_h , for $h = 1, \dots, k$, occurs in a run iff the trigger event e_s occurs in that run.*
2. *For every $h, g \leq k$ the relative timing of the responses is exactly the difference in the associated delta values: $t_h - t_g = \delta_h - \delta_g$*

Note how the new TTR problem definition generalizes the previous SR definition: A simultaneous response problem $\text{SR} = \langle e_s, \alpha_1, \dots, \alpha_k \rangle$ is simply an extremely tight TTR problem, where $\delta_1, \dots, \delta_k$ are all set to 0.

Intuitively, we expect that an agent i participating in a solution to a TTR task by performing response α_i at time t_i will know that agent j will perform its own response α_j at precisely $t_j = \delta_j - \delta_i + t_i$. We also expect agent j at t_j to know that i is carrying out α_i at t_i . Theorem 8 below shows that these individually specified knowledge states, and others that are derived from the TTR definition, add up to a node based common knowledge gain requirement.

Theorem 8. *Let $TTR = TTR = \langle e_s, \alpha_1: \delta_1, \alpha_2: \delta_2, \dots, \alpha_k: \delta_k \rangle$, and assume that TTR is solved in \mathcal{R} . Let $r \in \mathcal{R}$ be a run in which e_s occurs, and let $A = \{\alpha_1, \dots, \alpha_k\}$ be the set of nodes at which the responses are carried out in the run r (α_1 occurs at node α_1 , etc.). WLOG, let $\alpha_h = \langle i_h, t_h \rangle$ be the earliest node in A . Then $(\mathcal{R}, r) \models C_A \text{occurred}_{t_h}(e_s)$.*

Summing up, the TTR problem requires agents to be fully informed with respect to each other's response time - without the extra requirement for simultaneity. Since each of the agents, as it responds, knows the response times of all other agents - we end up with node based common knowledge, which in turn still requires that a broom node exist that can guarantee that all of its messages to the responding agents are delivered by the time these agents are set to respond.

7 Conclusions

This paper explores the implications of a new formalism for epistemic statements upon the notions of nested and common knowledge. It checks how these altered concepts interact with communication on the one hand, and coordination on the other - along the lines of [5, 4]. The new formalism is seen to allow for a decoupling of epistemic necessity and the past tense, in the sense that knowing that an event must occur or that an agent gains knowledge of a fact no longer entails that the occurrence, or the knowledge gain, have happened in the past. This, in turn, allows for a wider range of coordination tasks to be characterized in terms of the epistemic states that they necessitate. We define two such coordination tasks. The Weakly Timed and the Tightly Timed Response problems, extending our previous definitions of Ordered and Simultaneous Response [5, 4].

Our analysis of the relations between knowledge and coordination yields two valuable insights. First, when agents must respond within interrelated time bounds, as seen in the WTR problem, the crucial knowledge that they must gain before applying their assigned actions concerns those responses whose response times are *bounded from above* with respect to their own response. This allows them, if necessary, to delay their own action in order to conform with the problem requirements.

The enquiry into the node-based extension to common knowledge is even more rewarding. Much has been written about the relation between common knowledge and simultaneity. Formal analysis of this relation is given in [12, 11]. Other analyses implicitly rely upon simultaneity in accounting for common knowledge gain, using such concepts as *public announcements* [3]

or the *copresence heuristics* [9]. The analysis presented here sheds light on this issue by generalizing common knowledge and showing that the more general form corresponds to a temporally tight form of coordination. The previously established connection between common knowledge and simultaneity is, in fact, a particular instance of this more general connection. Recall by Lemma 1 that there is an embedding of the standard common knowledge $C_G\text{occurred}(e_s)$ at time t into the formula $C_A\text{occurred}_t(e_s) \in \mathcal{L}_1$ where all nodes in A are timed to t . The particular form of coordination corresponding to the embedded formula is tight coordination with all deltas set to 0—namely, simultaneity at time t .

We consider this paper as a point of departure for several lines of potential further research. First, we have only touched the surface of the required logical analysis for node based semantics. Completeness and tractability issues remain unknown, as well as possible variations on the node based theme that will chime in with ongoing research in the temporal logic and model checking communities. Second, although we have mainly been concerned with temporally oriented coordination, the analysis may also be instructive where other aspects of coordination are concerned. Chwe’s [8] analysis of the communication network required in order to bring about a revolt in a social setting is a case in point for further research. Finally, we have yet to study the implications of the WTR and TTR problems defined here in the context of distributed computing tasks, where timed coordination is often essential.

References

- [1] R. Alur and T. Henzinger. Logics and models of real-time: A survey. In *Real Time: Theory in Practice*, volume 600, pages 74–106. Springer-Verlag, 1992.
- [2] C. Areces and B. ten Cate. Hybrid logics. In P. Blackburn, F. Wolter, and J. van Benthem, editors, *Handbook of Modal Logics*. Elsevier, 2006.
- [3] A. Baltag, L. S. Moss, and S. Solecki. The logic of public announcements, common knowledge, and private suspicions. In *Proceedings of the 7th conference on Theoretical aspects of rationality and knowledge*, TARK ’98, San Francisco, CA, USA, 1998. Morgan Kaufmann Publishers Inc.
- [4] I. Ben-Zvi. *Causality, Knowledge and Coordination in Distributed Systems*. PhD thesis, Technion, Israel, 2011.

- [5] I. Ben-Zvi and Y. Moses. Beyond Lamport’s happened-before: On the role of time bounds in synchronous systems. In *DISC 2010*, pages 421–436, 2010.
- [6] I. Ben-Zvi and Y. Moses. On interactive knowledge with bounded communication (extended abstract). In *LOFT 2010*, 2010.
- [7] K. M. Chandy and J. Misra. How processes learn. *Distributed Computing*, 1(1):40–52, 1986.
- [8] M. S.-Y. Chwe. Communication and coordination in social networks. *Review of Economic Studies*, 67(1), 2000.
- [9] H. H. Clark and C. R. Marshall. Definite reference and mutual knowledge. In A. K. Joshi, B. L. Webber, and I. Sag, editors, *Elements of Discourse Understanding*, pages 10–63. Cambridge University Press, Cambridge, 1981.
- [10] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. Common knowledge revisited. In Y. Shoham, editor, *Theoretical Aspects of Rationality and Knowledge: Proc. Sixth Conference*, pages 283–298. Morgan Kaufmann, San Francisco, Calif., 1996.
- [11] R. Fagin, J. Y. Halpern, Y. Moses, and M. Y. Vardi. *Reasoning about Knowledge*. MIT Press, Cambridge, Mass., 2003.
- [12] J. Y. Halpern and Y. Moses. Knowledge and common knowledge in a distributed environment. *Journal of the ACM*, 37(3):549–587, 1990. A preliminary version appeared in *Proc. 3rd ACM Symposium on Principles of Distributed Computing*, 1984.
- [13] L. Lamport. Time, clocks, and the ordering of events in a distributed system. *Communications of the ACM*, 21(7):558–565, 1978.

A Proofs

This appendix contains all lemmas and major proofs necessary to support the claims of the paper. Note that the formal work leading up to Theorems 5 and 7 is non-trivial, and the interested reader would be better off reading the more detailed account presented in [5, 6, 4].

Lemma 1. *There exists a function $\text{ts} : \mathcal{L}_0 \times \text{Time} \rightarrow \mathcal{L}_1$ such that for every $\varphi \in \mathcal{L}_0$, time t , and $\varphi^t = \text{ts}(\varphi, t)$:*

$$(\mathcal{R}, r, t) \models \varphi \quad \text{iff} \quad (\mathcal{R}, r) \models \varphi^t.$$

Proof Proof is by structural induction on φ . We go over the clauses. Each clause is used to define ts , but it is also easy to see that based on the clause definition $(\mathcal{R}, r, t) \models \varphi$ iff $(\mathcal{R}, r) \models \varphi^t$.

- $\text{ts}(\text{occurred}(e), t) = \text{occurred}_t(e)$.
- $\text{ts}(K_i\psi, t) = K_{\langle i, t \rangle}\psi$.
- $\text{ts}(E_G\psi, t) = E_A\psi$ where $A = \{\langle j, t \rangle : j \in G\}$.
- $\text{ts}(C_G\psi, t) = C_A\psi$ where $A = \{\langle j, t \rangle : j \in G\}$.

■

Lemma 2.

- Both $K_{\langle i, t \rangle}$ and C_A are S5 modalities.
- $C_A\varphi \Rightarrow E_A(\varphi \wedge C_A\varphi)$ is valid.
- If $\mathcal{R} \models \varphi \Rightarrow E_A(\varphi \wedge \psi)$ then $\mathcal{R} \models \varphi \Rightarrow C_A\psi$

Proof

For S5 modality the $K_{\langle i, t \rangle}$ case is immediate. We focus on showing for C_A :

K $(C_A\varphi \wedge C_A(\varphi \Rightarrow \psi) \Rightarrow C_A\psi)$: Suppose that $(\mathcal{R}, r) \models C_A\varphi \wedge C_A(\varphi \Rightarrow \psi) \wedge \neg C_A\psi$ for some r, φ, ψ and A . Then there is a sequence $\alpha_1, \alpha_2, \dots, \alpha_n \in A$ such that $(\mathcal{R}, r) \not\models K_{\alpha_n} K_{\alpha_{n-1}} \dots K_{\alpha_1} \psi$. From $(\mathcal{R}, r) \models C_A\varphi$ and $(\mathcal{R}, r) \models C_A(\varphi \Rightarrow \psi)$ we obtain $(\mathcal{R}, r) \models K_{\alpha_n} K_{\alpha_{n-1}} \dots K_{\alpha_1} \varphi$ and $(\mathcal{R}, r) \models K_{\alpha_n} K_{\alpha_{n-1}} \dots K_{\alpha_1} (\varphi \Rightarrow \psi)$ respectively, and using the **K** axiom for the knowledge operator and trivial induction we get that $(\mathcal{R}, r) \models K_{\alpha_h} K_{\alpha_{h-1}} \dots K_{\alpha_1} \psi$ for all $h \leq n$, and in particular for $h = n$. This contradicts the assumption that $(\mathcal{R}, r) \models C_A\varphi \wedge C_A(\varphi \Rightarrow \psi) \wedge \neg C_A\psi$.

- T** ($C_A\varphi \Rightarrow \varphi$): Suppose that $(\mathcal{R}, r) \models C_A\varphi$ for some r, φ and A . Fix some $\langle i, t \rangle \in A$. From $(\mathcal{R}, r) \models C_A\varphi$ we get $(\mathcal{R}, r) \models K_{\langle i, t \rangle}\varphi$, and by definition of \models we conclude that $(\mathcal{R}, r') \models \varphi$ for all r' such that $r'_i(t) = r_i(t)$, and in particular for $r' = r$. Thus $(\mathcal{R}, r) \models \varphi$.
- 4** ($C_A\varphi \Rightarrow C_A C_A\varphi$): Suppose not. Then there exist r, A, φ such that $(\mathcal{R}, r) \models C_A\varphi$ and a sequence $\alpha_1, \alpha_2, \dots, \alpha_n \in A$ such that $(\mathcal{R}, r) \not\models K_{\alpha_n} K_{\alpha_{n-1}} \dots K_{\alpha_1} C_A\varphi$. Once again, there must be a sequence $\alpha'_1, \alpha'_2, \dots, \alpha'_m \in A$ such that

$$(\mathcal{R}, r) \not\models K_{\alpha_n} \dots K_{\alpha_1} K_{\alpha'_m} \dots K_{\alpha'_1} \varphi.$$

Yet this contradicts $(\mathcal{R}, r) \models C_A\varphi$.

- 5** ($\neg C_A\varphi \Rightarrow C_A \neg C_A\varphi$): Suppose not. Then there exist r, A, φ such that $(\mathcal{R}, r) \models \neg C_A\varphi$ and a sequence $\alpha_1, \alpha_2, \dots, \alpha_n \in A$ such that $(\mathcal{R}, r) \not\models K_{\alpha_n} K_{\alpha_{n-1}} \dots K_{\alpha_1} \neg C_A\varphi$. By definition of \models there exists r' s.t. $r'_{i_n}(t_n) = r_{i_n}(t_n)$ and where

$$(\mathcal{R}, r^n) \not\models K_{\alpha_{n-1}} K_{\alpha_{n-2}} \dots K_{\alpha_1} \neg C_A\varphi.$$

Similar argumentation provides us with runs r^h for all $1 \leq h < n$ such that $(\mathcal{R}, r^h) \not\models K_{\alpha_{h-1}} K_{\alpha_{h-2}} \dots K_{\alpha_1} \neg C_A\varphi$. In particular for $h = 1$ we get that $(\mathcal{R}, r^1) \not\models \neg C_A\varphi$, and hence $(\mathcal{R}, r^1) \models C_A\varphi$, and finally that $(\mathcal{R}, r^1) \models K_{\alpha_n} K_{\alpha_{n-1}} \dots K_{\alpha_1} C_A\varphi$. Yet by build we have (i) $r^1_{i_1}(t_1) = r^2_{i_1}(t_1)$, $r^2_{i_2}(t_2) = r^3_{i_2}(t_2)$, etc., giving us $(\mathcal{R}, r) \models C_A\varphi$, contradicting our assumption.

- N** (**If** $\models \varphi$ **then** $\models C_A\varphi$): We show by induction on h that for any h length sequence $\alpha_1, \alpha_2, \dots, \alpha_h \in A$ we have $\models K_{\alpha_h} K_{\alpha_{h-1}} \dots K_{\alpha_1} \varphi$. For $h = 0$ this is immediate from the assumption. Assume for h . Fix run r . Note that for every run r' s.t. $r_{i_h}(t_h) = r'_{i_h}(t_h)$ we have $(\mathcal{R}, r') \models K_{\alpha_{h-1}} \dots K_{\alpha_1} \varphi$ by the inductive hypothesis. Hence we get that $(\mathcal{R}, r) \models K_{\alpha_h} K_{\alpha_{h-1}} \dots K_{\alpha_1} \varphi$. As this is true for any r we conclude that $\models K_{\alpha_h} K_{\alpha_{h-1}} \dots K_{\alpha_1} \varphi$ as required. By definition of $C_A\varphi$ we get that $\models C_A\varphi$.

$C_A\varphi \Rightarrow E_A(\varphi \wedge C_A\varphi)$: Fix r, φ, A . From $(\mathcal{R}, r) \models C_A\varphi$ we get by definition of C_A that (i) $(\mathcal{R}, r) \models K_\alpha \varphi$ and (ii) $(\mathcal{R}, r) \models K_\alpha C_A\varphi$ for all $\alpha \in A$. Hence we also get (iii) $(\mathcal{R}, r) \models E_A\varphi$ and (iv) $(\mathcal{R}, r) \models E_A C_A\varphi$. Putting (iii) and (iv) together we get $(\mathcal{R}, r) \models E_A(\varphi \wedge C_A\varphi)$.

If $\mathcal{R} \models \varphi \Rightarrow E_A(\varphi \wedge \psi)$ then $\mathcal{R} \models \varphi \Rightarrow C_A\psi$: Suppose $\mathcal{R} \models \varphi \Rightarrow E_A(\varphi \wedge \psi)$. Fix a sequence $\alpha_1, \alpha_2, \dots, \alpha_k \in A$. We will now show by induction on $h \leq k$ that $\mathcal{R} \models \varphi \Rightarrow K_{\alpha_k} K_{\alpha_{k-1}} \dots K_{\alpha_1} \psi$. For $h = 1$ we have by assumption that $\mathcal{R} \models \varphi \Rightarrow E_A(\varphi \wedge \psi)$. We replace E_A by K_{α_1} , recalling that $\alpha_1 \in A$. We get $\mathcal{R} \models \varphi \Rightarrow K_{\alpha_1} \psi$. Assume for h and look at the case of $h + 1$. By assumption $\mathcal{R} \models \varphi \Rightarrow E_A(\varphi \wedge \psi)$ and hence $\mathcal{R} \models \varphi \Rightarrow K_{\alpha_{h+1}}(\varphi \wedge \psi)$. Fix run r . If $(\mathcal{R}, r) \not\models \varphi$ then $(\mathcal{R}, r) \models \varphi \Rightarrow K_{\alpha_{h+1}} K_{\alpha_h} \dots K_{\alpha_1} \psi$ and we are done. Else, we have that $(\mathcal{R}, r) \models E_A(\varphi \wedge \psi)$, and so $(\mathcal{R}, r) \models K_{\alpha_{h+1}}(\varphi \wedge \psi)$. Thus, we get that $(\mathcal{R}, r') \models \varphi$ for every r' such that $r_{i_{h+1}}(t_{h+1}) = r'_{i_{h+1}}(t_{h+1})$. By the inductive hypothesis we get that $(\mathcal{R}, r') \models K_{\alpha_h} \dots K_{\alpha_1} \psi$. By definition of \models we get that $(\mathcal{R}, r) \models K_{\alpha_{h+1}} K_{\alpha_h} \dots K_{\alpha_1} \psi$. We conclude that $\mathcal{R} \models \varphi \Rightarrow K_{\alpha_k} K_{\alpha_{k-1}} \dots K_{\alpha_1} \psi$. As this holds for any sequence of nodes in A we get that $\mathcal{R} \models \varphi \Rightarrow C_A\psi$ as required. ■

Definition 13 (Past cone). Fix $r \in \mathcal{R}$ and $\theta \in \mathbb{V}$. The past cone of θ in r , $\text{past}(r, \theta)$, is the set $\{\psi : \psi \rightsquigarrow \theta\}$.

Definition 14 (Agree upon). We say that two runs r and r' agree on the node $(i, t) \in \mathbb{V}$ if

1. $r_i(t) = r'_i(t)$,
2. the same external inputs and messages arrive at (i, t) in both runs, and
3. the same actions are performed by i at time t .

Lemma 3. Let $r \in \mathcal{R}$ and let $\theta \in \mathbb{V}$. Then there is a run $r' \in \mathcal{R}$ such that

1. $\text{past}(r', \theta) = \text{past}(r, \theta)$,
2. r' and r agree on all the nodes of $\text{past}(r', \theta)$.
3. the only nondeterministic events in r' occur at nodes of $\text{past}(r', \theta)$.

Lemma 4. If $\langle i, t \rangle \rightsquigarrow \langle j, t' \rangle$ then $t \leq t'$, with $t = t'$ holding only if $i = j$.

Definition 15 (Early delivery). When a message sent at $\langle i, t \rangle$ arrives at $\langle j, t' \rangle$ prior to the maximal allowed delay, i.e. when $t' < t + \max_{ij}$, then we say that an early delivery nondeterministic event has occurred.

Lemma 5. Fix a run r , and let $\theta \rightsquigarrow \theta'$. If $\theta \not\rightarrow \theta'$ then there is a node β such that $\theta \rightsquigarrow \beta \dashrightarrow \theta'$ and an early delivery occurs at β in r .

Lemma 6. Let e be the delivery of an external input that occurs at the node α_0 in $r \in \mathcal{R}$. If $(\mathcal{R}, r) \models K_{\alpha_1} \text{occurred}_{t_1}(e)$ then $\alpha_0 \rightsquigarrow \alpha_1$.

Theorem 5 (Knowledge Gain). Let $r \in \mathcal{R}$. Assume that e is an external input event occurring at α_0 in r .

If $(\mathcal{R}, r) \models K_{\alpha_k} K_{\alpha_{k-1}} \cdots K_{\alpha_1} \text{occurred}_{t_1}(e)$, then there is an uneven centipede for $\langle \alpha_0, \dots, \alpha_k \rangle$ in r .

Proof First note that $k \geq 1$ and $(\mathcal{R}, r) \models K_{\alpha_k} K_{\alpha_{k-1}} \cdots K_{\alpha_1} \text{occurred}_{t_1}(e)$ imply by the Knowledge Axiom that $(\mathcal{R}, r) \models K_{\alpha_k} \text{occurred}_{t_1}(e)$. It follows that $\alpha_0 \rightsquigarrow \alpha_k$ in r by Lemma 6. We prove the claim by induction on $k \geq 1$:

$k = 1$ As argued above, $\alpha_0 \rightsquigarrow \alpha_k$. Thus, $\alpha_0 \rightsquigarrow \alpha_1$ since $k = 1$, and so $\langle \alpha_0, \alpha_k \rangle$ is a (trivial) centipede for $\langle \alpha_0, \alpha_k \rangle$ in r .

$k \geq 2$ Assume inductively that the claim holds for $k - 1$. Moreover, assume that $(\mathcal{R}, r) \models K_{\alpha_k} K_{\alpha_{k-1}} \cdots K_{\alpha_1} \text{occurred}_{t_1}(e)$. Let r' be the run guaranteed by Lemma 3 to exist with respect to r , i_k and t_k . Recall from Lemma 3(1) that $\text{past}(r', \alpha_k) = \text{past}(r, \alpha_k)$. Thus, $\alpha_0 \rightsquigarrow \alpha_k$ in r implies that $\alpha_0 \rightsquigarrow \alpha_k$ in r' too. Moreover, by Lemma 3(2) we have that r and r' agree on the nodes of $\text{past}(r, \alpha_k)$, so in particular $r'_{i_k}(t_k) = r_{i_k}(t_k)$. Since $(\mathcal{R}, r) \models K_{\alpha_k} K_{\alpha_{k-1}} \cdots K_{\alpha_1} \text{occurred}_{t_1}(e)$ and $r'_{i_k}(t_k) = r_{i_k}(t_k)$, we have that

$$(\mathcal{R}, r', t') \models K_{\alpha_{k-1}} \cdots K_{\alpha_1} \text{occurred}_{t_1}(e).$$

By the inductive hypothesis there exists a centipede $\langle \alpha_0, \theta_1, \dots, \theta_{k-1} \rangle$ for $\langle \alpha_0, \dots, \alpha_{k-1} \rangle$ in r' . Let $c \geq 0$ be the minimal index for which $\theta_c \dashrightarrow \theta_h$ for all $h = c + 1, \dots, k - 1$. Clearly $c \leq k - 1$, since $\theta_{k-1} \dashrightarrow \theta_{k-1}$.

- If $c = 0$ then $\alpha_0 \dashrightarrow \theta_h \dashrightarrow \alpha_h$, and thus also $\alpha_0 \dashrightarrow \alpha_h$, for $h = 1, \dots, k - 1$. Since $\alpha_0 \rightsquigarrow \alpha_k$ in r , it follows that the tuple $\langle (\alpha_0)^{d-1}, \alpha_k \rangle$ (in which α_0 plays the role of the first $k - 1$ nodes) is a centipede for $\langle \alpha_0, \dots, \alpha_k \rangle$ in r .
- Otherwise, $c > 0$ and $\theta_{c-1} \rightsquigarrow \theta_c$ while $\theta_{c-1} \not\rightarrow \theta_c$. By Lemma 5 it follows that there exists a node β such that $\theta_{c-1} \rightsquigarrow \beta \dashrightarrow \theta_c$, and β is the site of an early receive in the run r' . By construction of r' , early receives can arrive only at nodes in $\text{past}(r', \alpha_k)$. It follows

that $\beta \rightsquigarrow \alpha_k$ in r' , and since $\text{past}(r, \alpha_k) = \text{past}(r', \alpha_k)$, we have that $\beta \rightsquigarrow \alpha_k$ in r too and hence also that $\text{past}(r, \beta) = \text{past}(r', \beta)$. It follows that in r

$$\alpha_0 \rightsquigarrow \theta_1 \rightsquigarrow \dots \rightsquigarrow \theta_{c-1} \rightsquigarrow \beta \rightsquigarrow \alpha_k.$$

Recall that \dashrightarrow depends only on the weighted communication network, which is the same in both r and r' . Thus, $\theta_j \dashrightarrow \alpha_j$ for all $0 < j \leq c-1$. Moreover, $\beta \dashrightarrow \theta_h \dashrightarrow \alpha_h$ and so $\beta \dashrightarrow \alpha_h$ for all $h = c, c+1, \dots, k-1$. It follows that $\langle \alpha_0, \theta_1, \dots, \theta_{c-1}, (\beta)^{k-c}, \alpha_k \rangle$ is a centipede for $\langle \alpha_0, \dots, \alpha_k \rangle$ in r .

It follows that a centipede for $\langle \alpha_0, \dots, \alpha_k \rangle$ in r is guaranteed to exist in all cases, as claimed.

$\square_{\text{Theorem 5}}$

Theorem 6. *Let $\text{WTR} = \langle e_s, \alpha_1 : \delta_1, \dots, \alpha_{k-1} : \delta_{k-1}, \alpha_k \rangle$ be an instance of WTR, and assume that WTR is solved in the system \mathcal{R} . Let $r \in \mathcal{R}$ be a run in which e_s occurs. For each $h \leq k$, let $\alpha_h = \langle i_h, t_h \rangle$ be the node at which response α_h gets performed, and let*

$$\beta_h^k = \langle i_h, t_k - \sum_{j=h}^{k-1} \delta_j \rangle.$$

Then

$$(\mathcal{R}, r) \models K_{\alpha_k} K_{\beta_{k-1}^k} \dots K_{\beta_1^k} \text{occurred}_{t_1^k}(e_s).$$

Proof We prove the theorem by induction on $h \leq k$.

$h = 1$: By definition of r , α_1 gets performed at α_1 . Since performing a local action is written in the agent's local state (and hence known to the agent), and since it is always performed no sooner than the triggering event e_s , we have $(\mathcal{R}, r) \models K_{\alpha_1} \text{occurred}_{t_1}(e_s)$. If $h = k = 1$ then we are done. Else, as $t_1^k \geq t_1$ and as agents have perfect recall, we get $(\mathcal{R}, r) \models K_{\beta_1^k} \text{occurred}_{t_1^k}(e_s)$.

$h > 1$: Assume that $(\mathcal{R}, r) \models K_{\beta_{h-1}^k} \dots K_{\beta_1^k} \text{occurred}_{t_1^k}(e_s)$. As response α_h gets performed at α_h in r we have $(\mathcal{R}, r) \models K_{\alpha_h} \text{occurred}_{t_h^h}(a_h)$. Since WTR is solved in the system \mathcal{R} we have that in every run r' such that $r_{i_h}(t_h) = r'_{i_h}(t_h)$ response α_{h-1} gets performed at some t_{h-1} such

that $t_{h-1} \leq t_{h-1}^h$. Note that every system that solves WTR also solves the sub problem $\text{WTR}' = \langle e_s, \alpha_1 : \delta_1, \dots, \alpha_{h-2} : \delta_{h-2}, \alpha_{h-1} \rangle$. Based on the inductive hypothesis we obtain that

$$(\mathcal{R}, r') \models K_{\alpha_{h-1}} K_{\beta_{h-2}^k} \cdots K_{\beta_1^k} \text{occurred}_{t_1^k}(e_s).$$

From $t_{h-1}^k \geq t_{h-1}$, and based on perfect recall, we derive that

$$(\mathcal{R}, r') \models K_{\beta_{h-1}^k} K_{\beta_{h-2}^k} \cdots K_{\beta_1^k} \text{occurred}_{t_1^k}(e_s).$$

By our choice of runs r' and the definition of \models this gives us

$$(\mathcal{R}, r') \models K_{\alpha_h} K_{\beta_{h-1}^k} K_{\beta_{h-2}^k} \cdots K_{\beta_1^k} \text{occurred}_{t_1^k}(e_s).$$

If $h = k$ then we are done. Else, once again deploying $t_h^k \geq t_h$ we conclude that

$$(\mathcal{R}, r') \models K_{\beta_h^k} K_{\beta_{h-1}^k} K_{\beta_{h-2}^k} \cdots K_{\beta_1^k} \text{occurred}_{t_1^k}(e_s),$$

and we are done.

$\square_{\text{Theorem 6}}$

Theorem 7. Let $A \subseteq \mathbb{V}$ with $\langle j_k, t'_k \rangle$ being the earliest node ($t'_k \leq t'_h$ for all $\langle j_h, t'_h \rangle \in A$), and let $r \in \mathcal{R}$. Assume that e is an external input event at α_0 in r . If $(\mathcal{R}, r) \models C_A(\text{occurred}_{t'_k}(e))$, then there is a uneven broom $\hat{\theta}$ for $\langle \alpha_0, A \rangle$ in r .

Proof Assume the notations and conditions of the theorem. Denote $A = \{\alpha_1, \dots, \alpha_k\}$ and $d = t'_k - t_0$, the time difference between the occurrence of e and the latest node in A . Since $(\mathcal{R}, r) \models C_A(\text{occurred}_{t'_k}(e))$ we have by definition of common knowledge that

$(\mathcal{R}, r) \models E_A^{k(d+1)} \text{occurred}_{t'_k}(e)$. In particular, this implies that

$$(\mathcal{R}, r) \models (K_{\alpha_k} \cdots K_{\alpha_1})^{d+1} \text{occurred}_{t'_k}(e),$$

where $(K_{\alpha_k} \cdots K_{\alpha_1})^{d+1}$ stands for $d+1$ consecutive copies of $K_{\alpha_k} \cdots K_{\alpha_1}$. By the Knowledge Gain Theorem 5, there is a corresponding centipede $\sigma = \langle \theta_0, \theta_1, \dots, \theta_{k(d+1)} \rangle$ in r . Denote $\theta_h = (i_h, t_h)$ for all $0 \leq h \leq k(d+1)$. Recall that, by definition, $\theta_h \rightsquigarrow \theta_{h+1}$ holds for all $h < k(d+1)$. By Lemma 4 we obtain that if $\theta_h \neq \theta_{h+1}$ then $t_h < t_{h+1}$. It follows that there can be at

most $d + 1$ distinct nodes $\theta'_1 \rightsquigarrow \theta'_2 \rightsquigarrow \dots \rightsquigarrow \theta'_\ell$ in σ . Every θ'_h represents a segment $\theta_x, \dots, \theta_{x+s}$ of the nodes in σ . By the pigeonhole principle, one of the θ' nodes must represent a segment consisting of at least k of the original θ s in σ . Denoting this node by $\hat{\theta}$, we obtain that $\hat{\theta} \dashrightarrow \alpha_h$ for every $\alpha_h \in A$. Moreover, by definition of the centipede and transitivity of \rightsquigarrow we have that $\alpha_0 \rightsquigarrow \hat{\theta}$. It follows that $\hat{\theta}$ is a centibroom for $\langle \alpha_0, G \rangle$ in r . $\square_{\text{Theorem 7}}$

Theorem 8. *Let $\text{TTR} = \langle e_s, \alpha_1 : \delta_1, \dots, \alpha_k : \delta_k \rangle$, and assume that TTR is solved in \mathcal{R} . Let $r \in \mathcal{R}$ be a run in which e_s occurs, and let $A = \{\alpha_1, \dots, \alpha_k\}$ be the set of nodes at which the responses are carried out in the run r (α_1 occurs at node $\alpha_1 = \langle i_1, t_1 \rangle$, etc.). Let $\alpha' = \langle i', t' \rangle$ be the earliest node in A . Then $(\mathcal{R}, r) \models C_A \text{occurred}_{t'}(e_s)$.*

Proof Fix $h, g \in \{1..k\}$. We first show that

$$\mathcal{R} \models \text{occurred}_{t_h}(a_h) \Rightarrow E_A(\text{occurred}_{t_h}(a_h) \wedge \text{occurred}_{t'}(e_s)).$$

Choose r' such that $(\mathcal{R}, r') \models \text{occurred}_{t_h}(a_h)$. Note that since TTR is solved in \mathcal{R} and since $t_h - t_g = \delta_h - \delta_g$ in every triggered run by problem definition, we get $(\mathcal{R}, r') \models \text{occurred}_{t_h}(a_h) \leftrightarrow \text{occurred}_{t_g}(a_g)$. Since performing a local action is written, at least for the current round, in the agent's local state (and hence known to the agent), we obtain that $(\mathcal{R}, r') \models K_{\alpha_g} \text{occurred}_{t_g}(a_g)$, and hence also $(\mathcal{R}, r') \models K_{\alpha_g} \text{occurred}_{t_h}(a_h)$. Since h is arbitrarily chosen in $\{1..k\}$, node α_g knows this for all responding nodes, and in particular for the earliest responding node $\alpha' = \langle i', t' \rangle$. As responses always occur no sooner than the trigger, we get that $(\mathcal{R}, r') \models K_{\alpha_g} \text{occurred}_{t'}(e_s)$. Putting these results together we conclude that $(\mathcal{R}, r') \models K_{i_g}(\text{occurred}_{t_h}(a_h) \wedge \text{occurred}_{t'}(e_s))$. Since g is arbitrarily chosen in $\{1..k\}$, we get $(\mathcal{R}, r') \models E_A(\text{occurred}_{t_h}(a_h) \wedge \text{occurred}_{t'}(e_s))$, from which it follows that

$$(\mathcal{R}, r') \models \text{occurred}_{t_h}(a_h) \Rightarrow E_A(\text{occurred}_{t_h}(a_h) \wedge \text{occurred}_{t'}(e_s))$$

by our choice of r' . As false antecedents imply anything, we also get

$$(\mathcal{R}, r'') \models \text{occurred}_{t_h}(a_h) \Rightarrow E_A(\text{occurred}_{t_h}(a_h) \wedge \text{occurred}_{t'}(e_s))$$

in runs r'' where response α_h does not occur (non triggered runs) or where it occurs at a later time than t_h . We thus conclude that $\mathcal{R} \models \text{occurred}_{t_h}(a_h) \Rightarrow E_A(\text{occurred}_{t_h}(a_h) \wedge \text{occurred}_{t'}(e_s))$.

Recall the Knowledge Induction Rule in Lemma 2, that provides us with $\mathcal{R} \models \varphi \Rightarrow C_A \psi$ from $\mathcal{R} \models \varphi \Rightarrow E_A(\varphi \wedge \psi)$. Setting $\varphi = \text{occurred}_{t_h}(a_h)$ and

$\psi = \text{occurred}_{t'}(e_s)$ we apply the rule, and based on the above result obtain
 $\mathcal{R} \models \text{occurred}_{t_h}(a_h) \Rightarrow C_A \text{occurred}_{t'}(e_s)$. We conclude by taking notice that
 $(\mathcal{R}, r) \models \text{occurred}_{t_h}(a_h)$ by assumption, and hence also $(\mathcal{R}, r) \models C_A \text{occurred}_{t'}(e_s)$.
 $\square_{\text{Theorem 8}}$